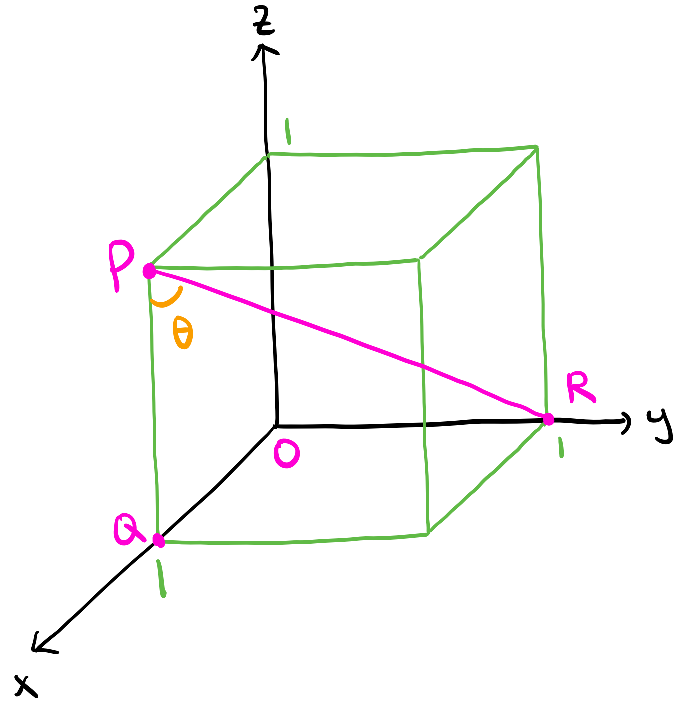
1. (10 points) Let θ be the angle between an edge of the cube and the long diagonal from an endpoint of the same edge to the opposite corner of the cube (this long diagonal contains the center of the cube). Make a sketch of the cube, the edge, and the diagonal and then use it to find $\cos\theta$.



$P = (1,0,1)$

$Q = (1,0,0)$

$R = (0,1,0)$

$$\cos\theta = \frac{\vec{PQ}\cdot\vec{PR}}{|\vec{PQ}|\,|\vec{PR}|}$$

$\vec{PQ} = (0,0,-1)$

$\vec{PR} = (-1,1,-1)$

$$\Rightarrow \vec{PQ}\cdot\vec{PR} = 0\cdot(-1) + 0\cdot 1 + (-1)\cdot(-1) = 1$$

$$|\vec{PQ}| = \sqrt{0^2+0^2+(-1)^2} = 1$$

$$|\vec{PR}| = \sqrt{(-1)^2+1^2+(-1)^2} = \sqrt{3}.$$

$$\Rightarrow \cos\theta = \frac{1}{1\cdot\sqrt{3}} = \boxed{\frac{1}{\sqrt{3}}}$$

<u>Note</u> You get the same answer with other choices of the edge and the diagonal.

(e.g. $\vec{OR}$ and $\vec{PR}$)

2. Consider the plane $x + 2y - 2z = 3$ and the point $P = (0, -1, 2)$.

a) (3 points) Find the parametric equation of the line through point P that is orthogonal to the plane.

A direction vector $\vec{v}$ of the line is a normal vector of the plane $x+2y-2z=3$.

$\Rightarrow \vec{v} = (1, -2, 2)$

The line is then parametrized by

$\vec{\ell}(t) = \boxed{(0+1\cdot t,\ -1+2t,\ 2-2t)}$

b) (3 points) Find the point of intersection of that line and the plane. This is the point on the plane closest to P.

On the line: $x = 0+1\cdot t = t$, $y = -1+2t$, $z = 2-2t$.

The plane equation: $x+2y-2z=3$

$\Rightarrow t + 2(-1+2t) - 2(2-2t) = 3$

$\rightsquigarrow t - 2 + 4t - 4 + 4t = 3$

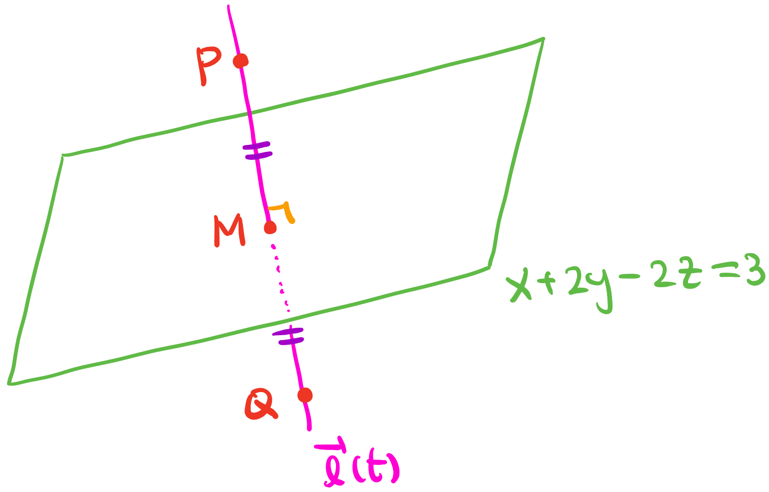
$\rightsquigarrow 9t = 9 \rightsquigarrow t = 1$.

The intersection point is

$\vec{\ell}(1) = (1, -1+2\cdot 1, 2-2\cdot 1)$

$= \boxed{(1, 1, 0)}$

c) (4 points) Find the point Q such that Q is the mirror image (or reflection) of P with respect to the plane.



Set $Q=(a,b,c)$.

The intersection of the line and the plane is also the midpoint of PQ.

$$\Rightarrow (1,1,0) = \left(\frac{0+a}{2}, \frac{-1+b}{2}, \frac{2+c}{2}\right)$$

$$\rightsquigarrow 1=\frac{a}{2}, \quad 1=\frac{-1+b}{2}, \quad 0=\frac{2+c}{2}$$

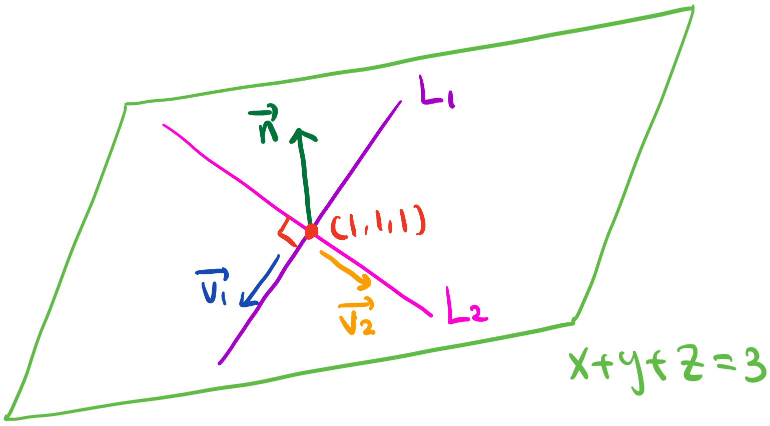
$$\rightsquigarrow a=2, \ b=3, \ c=-2$$

$$\Rightarrow Q = \boxed{(2,3,-2)}$$

<u>Note</u> Alternatively, since you have $t=0$ at P and $t=1$ at M, you get

$$Q=\vec{l}(2) = (2, -1+2\cdot 2, 2-2\cdot 2) = (2,3,-2).$$

3. Consider the plane $x + y + z = 3$ and the line L_1 given by $(x, y, z) = (1 + 3t, 1 - 2t, 1 - t)$. The line L_1 lies on the plane. Find the parametric equation of another line L_2 that contains the point $(1, 1, 1)$ like L_1 and also lies on the plane but is orthogonal to L_1.



A normal vector of the plane $x+y+z=3$ is

$\vec{n} = (1,1,1)$.

L_1 is parametrized by $\vec{\ell_1}(t) = (1+3t,\ 1-2t,\ 1-t)$

$\Rightarrow$ A direction vector is $\vec{v_1} = (3,-2,-1)$.

A direction vector $\vec{v_2}$ of L_2 is perpendicular to

both $\vec{n}$ and $\vec{v_1}$.

$\Rightarrow \vec{v_2} = \vec{n} \times \vec{v_1} = (1,1,1) \times (3,-2,-1) = (1,4,-5)$.

$\Rightarrow$ L_2 is parametrized by

$\vec{\ell_2}(t) = \boxed{(1+1\cdot t,\ 1+4t,\ 1-5t)}$

<u>Note</u> You can also take $\vec{v_2} = \vec{v_1} \times \vec{n} = (-1,-4,5)$

to get another parametrization

$\vec{\ell_2}(t) = (1-1\cdot t,\ 1-4t,\ 1+5t)$

4. In this problem, we consider a particle moving on a helix.

a) (2 points) Suppose the position vector of a particle as a function of time t is given by $\mathbf{r}(t) = (\cos t, \sin t, 3t)$. Find the speed of the particle. (Speed is the magnitude of the velocity.)

Velocity : $\vec{r}'(t) = (-\sin t, \cos t, 3)$

Speed : $|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 3^2} = \boxed{\sqrt{10}}$

b) (3 points) The position vector of another particle moving on the same helix at a different speed is given by $\mathbf{r}(t) = (\cos 2t, \sin 2t, 6t)$. Find the magnitude of its acceleration.

Velocity : $\vec{r}'(t) = (-2\sin(2t), 2\cos(2t), 6)$

Acceleration : $\vec{r}''(t) = (-4\cos(4t), -4\sin(4t), 0)$

$\Rightarrow |\vec{r}''(t)| = \sqrt{16\cos^2(4t) + 16\sin^2(4t) + 0} = \boxed{4}$

c) (5 points) A particle moves on the same helix with constant speed equal to v. Find the magnitude of its acceleration.

The particle in (a) moves with speed $\sqrt{10}$.

The particle with speed v moves $\frac{v}{\sqrt{10}}$ times faster.

$\Rightarrow$ The position at time t is

$$\vec{r}(t) = \left(\cos\left(\frac{v}{\sqrt{10}}t\right), \sin\left(\frac{v}{\sqrt{10}}t\right), \frac{3v}{\sqrt{10}}t\right)$$

$$\Rightarrow \vec{r}'(t) = \left(-\frac{v}{\sqrt{10}}\sin\left(\frac{v}{\sqrt{10}}t\right), \frac{v}{\sqrt{10}}\cos\left(\frac{v}{\sqrt{10}}t\right), \frac{3v}{\sqrt{10}}\right).$$

$$\Rightarrow \vec{r}''(t) = \left(-\frac{v^2}{10}\cos\left(\frac{v}{\sqrt{10}}t\right), -\frac{v^2}{10}\sin\left(\frac{v}{\sqrt{10}}t\right), 0\right)$$

$$\Rightarrow |\vec{r}''(t)| = \sqrt{\frac{v^4}{100}\cos^2\left(\frac{v}{\sqrt{10}}t\right) + \frac{v^4}{100}\sin^2\left(\frac{v}{\sqrt{10}}t\right) + 0} = \boxed{\frac{v^2}{10}}$$

5. Suppose $z = f(x, y)$ and $x = u^2 - v^2$, $y = u^2 + v^2$.

a) (5 points) Find $\frac{\partial z}{\partial u}$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial u} = f_x \cdot \frac{\partial x}{\partial u} + f_y \cdot \frac{\partial y}{\partial u}$$

chain rule

$$\frac{\partial x}{\partial u} = \frac{\partial}{\partial u}(u^2 - v^2) = 2u, \quad \frac{\partial y}{\partial u} = \frac{\partial}{\partial u}(u^2 - v^2) = 2u \qquad (*)$$

$$\Rightarrow \frac{\partial z}{\partial u} = \boxed{f_x \cdot 2u + f_y \cdot 2u}$$

b) (5 points) Find $\frac{\partial^2 z}{\partial u^2}$.

$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}(2uf_x + 2uf_y)$$

(a)

$$= 2f_x + 2u\frac{\partial f_x}{\partial u} + 2f_y + 2u\frac{\partial f_y}{\partial u}$$

$$\frac{\partial f_x}{\partial u} = \frac{\partial f_x}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f_x}{\partial y} \cdot \frac{\partial y}{\partial u} = f_{xx} \cdot 2u + f_{xy} \cdot 2u$$

chain rule $\qquad (*)$

$$\frac{\partial f_y}{\partial u} = \frac{\partial f_y}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f_y}{\partial y} \cdot \frac{\partial y}{\partial u} = f_{yx} \cdot 2u + f_{yy} \cdot 2u$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2} = 2f_x + 2u(2uf_{xx} + 2uf_{xy}) + 2f_y + 2u(2uf_{yx} + 2uf_{yy})$$

$$= \boxed{2f_x + 2f_y + 4u^2 f_{xx} + 8u^2 f_{xy} + 4u^2 f_{yy}}$$

$f_{xy} = f_{yx}$

6. In each of the parts below, evaluate the partial derivative using implicit differentiation.

a) (3 points) $x^2 + y^2 + z^2 = 1$. Find $\frac{\partial z}{\partial x}$.

$x^2+y^2+z^2-1=0$.

Set $f(x,y,z) = x^2+y^2+z^2-1$.

$\Rightarrow f_x = 2x, \quad f_z = 2z$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{2x}{2z} = \boxed{-\frac{x}{z}}$$

b) (3 points) $r^2 = x^2 + y^2 + z^2$. Find $\frac{\partial r}{\partial x}$.

$r^2-x^2-y^2-z^2=0$

Set $f(x,y,z,r) = r^2-x^2-y^2-z^2$.

$\Rightarrow f_x = -2x, \quad f_r = 2r$

$$\Rightarrow \frac{\partial r}{\partial x} = -\frac{f_x}{f_r} = -\frac{-2x}{2r} = \boxed{\frac{x}{r}}$$

* The implicit function theorem works for any number of variables

c) (4 points) $z^2 - z(x^2 + y^2) + xy = 0$. Find $\frac{\partial z}{\partial x}$.

Set $f(x,y,z) = z^2 - z(x^2+y^2) + xy$

$\Rightarrow f_x = -2zx + y, \quad f_z = 2z - x^2 - y^2$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{f_x}{f_z} = -\frac{-2zx+y}{2z-x^2-y^2} = \boxed{\frac{2zx-y}{2z-x^2-y^2}}$$

7. Consider the paraboloid $z = x^2 + y^2$ and the point $P = (2, -1, 5)$.

a) (4 points) Find the tangent plane to the paraboloid at the point P.

The paraboloid $z = x^2+y^2$ is the graph of $f(x,y) = x^2+y^2$.

The tangent plane equation at $P = (2,-1,5)$ is

$$z = f(2,-1) + f_x(2,-1)(x-2) + f_y(2,-1)(y+1).$$

$$f_x = 2x \rightsquigarrow f_x(2,-1) = 4.$$

$$f_y = 2y \rightsquigarrow f_y(2,-1) = -2$$

$$\Rightarrow \boxed{z = 5 + 4(x-2) - 2(y+1)}$$

b) (2 points) Find the parametric equation of the normal line at P.

The tangent plane equation can be written as

$$4(x-2) - 2(y+1) - (z-5) = 0.$$

A direction vector $\vec{v}$ of the normal line is

a normal vector of the tangent plane.

$$\Rightarrow \vec{v} = (4, -2, -1)$$

The normal line at $P = (2,-1,5)$ is parametrized by

$$\vec{l}(t) = \boxed{(2+4t, -1-2t, 5-t)}$$

c) (4 points) The plane that contains the normal line and the origin $(0, 0, 0)$ intersects the paraboloid along a curve. Find the parametric equation of this curve.

The normal line contains two points

$P = \vec{l}(0) = (2, -1, 5)$, $Q = \vec{l}(1) = (6, -3, 4)$

A normal vector of the plane is

$\vec{OP} \times \vec{OQ} = (2, -1, 5) \times (6, -3, 4) = (11, 22, 0)$

The plane equation is

$11(x-0) + 22(y-0) + 0 \cdot (z-0) = 0$

$\rightsquigarrow x + 2y = 0$.

The intersection of the plane and the paraboloid is given by the equations $x + 2y = 0$ and $z = x^2 + y^2$.

Set $y = t \Rightarrow \begin{cases} x = -2y = -2t. \\ z = x^2 + y^2 = (-2t)^2 + t^2 = 5t^2. \end{cases}$

The intersection is parametrized by

$\vec{r}(t) = \boxed{(-2t, t, 5t^2)}$

<u>Note</u> You can also set $x = t$ and get a different parametrization $\left(t, -\frac{t}{2}, \frac{5t^2}{4}\right)$.

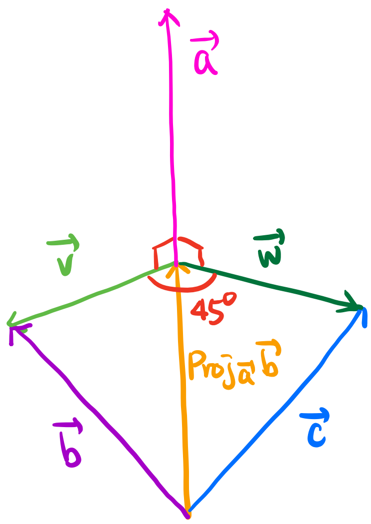
8. Suppose $\mathbf{a} = (1,1,1)$ and $\mathbf{b} = (1,1,-1)$ are two vectors.

a) (3 points) Find the projection of $\mathbf{b}$ along $\mathbf{a}$.

$$\mathrm{Proj}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\vec{a} = \frac{1\cdot1+1\cdot1+1\cdot(-1)}{1^2+1^2+1^2}(1,1,1) = \boxed{\frac{1}{3}(1,1,1)}$$

b) (7 points) Suppose $\mathbf{b}$ is rotated with $\mathbf{a}$ as the axis of rotation by 45^o or $\frac{\pi}{4}$ (with the sense of rotation determined by the right hand rule). Find the resulting vector.

Extremely tricky !

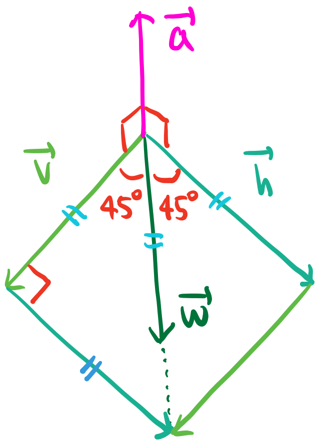


Let $\vec{c}$ be the resulting vector.

Set $\vec{v} = \vec{b} - \mathrm{Proj}_{\vec{a}}\vec{b} = \left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right)$.

Take $\vec{w}$ to be the vector resulting from rotating $\vec{v}$ about $\vec{a}$ by $45°$.

$\Rightarrow \vec{c} = \mathrm{Proj}_{\vec{a}}\vec{b} + \vec{w}$.



To find $\vec{w}$, we rotate $\vec{v}$ about $\vec{a}$ by $90°$ to get a vector $\vec{h}$.

$\Rightarrow \vec{h}$ is orthogonal to both $\vec{a}$ and $\vec{v}$ with $|\vec{h}| = |\vec{v}|$.

computation

$$\Rightarrow \vec{h} = \frac{|\vec{v}|}{|\vec{a}\times\vec{v}|}(\vec{a}\times\vec{v}) = \frac{1}{\sqrt{3}}(-2,2,0).$$

Since $\vec{v}$ and $\vec{h}$ form a square, $\vec{v}+\vec{h}$ is in the direction of $\vec{w}$ with $|\vec{v}+\vec{h}| = \sqrt{2}|\vec{v}| = \sqrt{2}|\vec{w}|$.

$$\Rightarrow \vec{w} = \frac{1}{\sqrt{2}}(\vec{v}+\vec{h}) = \frac{1}{\sqrt{2}}\left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right) + \frac{1}{\sqrt{6}}(-2,2,0)$$

$$\Rightarrow \vec{c} = \boxed{\frac{1}{3}(1,1,1) + \frac{1}{\sqrt{2}}\left(\frac{2}{3}, \frac{2}{3}, -\frac{4}{3}\right) + \frac{1}{\sqrt{6}}(-2,2,0)}$$